Linear Algebra and its Applications 431 (2009) 783–791

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

The generalized Drazin inverse with commutativity up to a factor in a Banach algebra

Dragana S. Cvetković-Ilić

Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, Višegradska 33, 18000 Niš, Serbia

A R T I C L E   I N F O

Article history:
Received 2 March 2009
Accepted 23 March 2009
Available online 25 April 2009

Submitted by R.A. Brualdi

AMS classification:
46H05
47A05
47A55

Keywords:
Generalized Drazin inverse
Commutativity
Banach algebra

A B S T R A C T

In this paper, we investigate an explicit representation of the generalized Drazin inverse $(a \pm b)^{\mathrm{d}}$ in terms of a, a^{d}, b and b^{d} under the condition $ab = \lambda ba$ or $ab = aba$ and extend to Banach algebras recent results of C.Y. Deng.

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with the unit 1. By $\mathcal{A}^{-1}$, $\mathcal{A}^{\mathrm{nil}}$ and $\mathcal{A}^{\mathrm{qnil}}$, we denote the sets of all invertible, nilpotent and quasinilpotent elements in $\mathcal{A}$, respectively. Let us recall that the Drazin inverse of $a \in \mathcal{A}$ [5] is the element $x \in \mathcal{A}$ (denoted by a^D) which satisfies

$$xax = x, \quad ax = xa, \quad a^{k+1}x = a^k, \tag{1}$$

for some nonnegative integer k. The least such k is the index of a, denoted by $\mathrm{ind}(a)$. When $\mathrm{ind}(a) = 1$, the Drazin inverse a^D is called the group inverse and it is denoted by $a^{\#}$. The conditions from (1) are equivalent to

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathcal{A}^{\mathrm{nil}}. \tag{2}$$

E-mail addresses: dragana@pmf.ni.ac.rs, gagamaka@ptt.rs

0024-3795/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2009.03.027

The concept of the generalized Drazin inverse in a Banach algebra was introduced by Koliha [10]. The condition $a - a^2x \in \mathcal{A}^{\mathrm{nil}}$ from (2) was replaced by the condition $a - a^2x \in \mathcal{A}^{\mathrm{qnil}}$. Hence, the generalized Drazin inverse of a is the element $x \in \mathcal{A}$ (written a^{d}) which satisfies

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathcal{A}^{\mathrm{qnil}}. \tag{3}$$

The Drazin index $\mathrm{ind}(a)$ of a is the nilpotency index of $a - a^2b$ if $a - a^2b \in \mathcal{A}^{\mathrm{nil}}$ and $\mathrm{ind}(a) = \infty$, otherwise.

We mention that an alternative definition of the generalized Drazin inverse in a normed algebra and a ring is also given in [7–9]:

Definition 1.1 (*Definition* 7.5.2 [8]). An element a of a normed algebra A is quasipolar if there exists idempotent $p \in A$ such that

$$ap = pa, \quad p \in (Aa) \cap (aA) \quad \text{and} \quad a(1-p) \in \mathcal{A}^{\mathrm{qnil}}. \tag{4}$$

Theorem 1.1 (Theorem 7.5.3 [8]). *If A is a normed algebra and $a \in A$ is quasipolar, then the idempotent p of Definition 1.1 is unique, lies in* $\mathrm{comm}^2(a)$ *and there is the unique $b \in A$ with*

$$ab = ba = p, \quad b = bp = pb.$$

Furthermore, $b \in \mathrm{comm}^2(a)$ and we called it the generalized Drazin inverse of a.

In [9] the definition of quasipolar is extended to general rings:

Definition 1.2 (*Definition* 5 [9]). An element a of a ring A is quasipolar if there exists $b \in A$ such that

$$b \in \mathrm{comm}^2(a), \quad ab = (ab)^2, \quad a(1-ab) \in \mathcal{A}^{\mathrm{qnil}}, \tag{5}$$

where $\mathrm{comm}^2(a)$ denotes the double commutant of a. An element b satisfying (5) and $b = ab^2$ is called a Drazin inverse of a.

The other version of the conditions (4) is given by Lemma 2.4 [10]:

Theorem 1.2. *In a ring A with unit, an element a has a Drazin inverse a^{d} if and only if there is an idempotent p commuting with a such that*

$$ap \in \mathcal{A}^{\mathrm{qnil}} \quad \text{and} \quad a + p \in \mathcal{A}^{-1}.$$

The Drazin inverse a^{d} is unique and is given by

$$a^{\mathrm{d}} = (a+p)^{-1}(1-p).$$

These two concepts of generalized Drazin inverse are equivalent in the case when the ring is actually a complex Banach algebra with the unit. It is well-known that a^{d} is unique whenever it exists [10].

The set $\mathcal{A}^{\mathrm{d}}$ consists of all $a \in \mathcal{A}$ such that a^{d} exists.

Let $a \in \mathcal{A}$ and let $p \in \mathcal{A}$ be an idempotent ($p = p^2$). Then we write

$$a = pap + pa(1-p) + (1-p)ap + (1-p)a(1-p)$$

and use the notations

$$a_{11} = pap, \quad a_{12} = pa(1-p), \quad a_{21} = (1-p)ap, \quad a_{22} = (1-p)a(1-p).$$

Every idempotent $p \in \mathcal{A}$ induces a representation of an arbitrary element $a \in \mathcal{A}$ given by the following matrix:

$$a = \begin{bmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p. \tag{6}$$

Let a^π be the spectral idempotent of a corresponding to $\{0\}$. It is well-known that $a \in \mathcal{A}^{\mathrm{d}}$ can be represented in the following matrix form:

$$a = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_p,$$

relative to $p = aa^{\mathrm{d}} = 1 - a^\pi$, where a_{11} is invertible in the algebra $p\mathcal{A}p$ and a_{22} is quasinilpotent element of the algebra $(1-p)\mathcal{A}(1-p)$. Using that representation, the Drazin inverse of a is given by

$$a^{\mathrm{d}} = \begin{bmatrix} a_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p,$$

where a_{11}^{-1} is the invertible element of a_{11} in subalgebra $p\mathcal{A}p$.

2. Results

The following result is proved in [11] for matrices, extended in [4] for a bounded linear operators and in [1] for arbitrary elements in a Banach algebra.

Theorem 2.1. *Let $x \in \mathcal{A}$ and let*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p \quad \text{and} \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-p}$$

relative to the idempotent $p \in \mathcal{A}$.

(1) *If $a \in (p\mathcal{A}p)^{\mathrm{d}}$ and $b \in ((1-p)\mathcal{A}(1-p))^{\mathrm{d}}$, then x, y are generalized Drazin invertible and*

$$x^{\mathrm{d}} = \begin{bmatrix} a^{\mathrm{d}} & u \\ 0 & b^{\mathrm{d}} \end{bmatrix}_p, \quad y^{\mathrm{d}} = \begin{bmatrix} b^{\mathrm{d}} & 0 \\ u & a^{\mathrm{d}} \end{bmatrix}_{1-p}, \tag{7}$$

where $u = \sum_{n=0}^{\infty}(a^{\mathrm{d}})^{n+2}cb^nb^\pi + \sum_{n=0}^{\infty}a^\pi a^n c(b^{\mathrm{d}})^{n+2} - a^{\mathrm{d}}cb^{\mathrm{d}}$.

(2) *If $x \in \mathcal{A}^{\mathrm{d}}$ and $a \in (p\mathcal{A}p)^{\mathrm{d}}$, then $b \in ((1-p)\mathcal{A}(1-p))^{\mathrm{d}}$ and x^{d} is given by (7).*

Next lemmas will be very useful in proving our main results:

Lemma 2.1. *Let $a, b \in \mathcal{A}$, $ab = \lambda ba$ and $ab \neq 0$. Then*

(1) *There is equality: $(ab)^n = \lambda^{-\frac{(n-1)n}{2}}a^nb^n = \lambda^{\frac{n(n+1)}{2}}b^na^n$.*

(2) *If in particular a is invertible, then*

$$(a^{-1}b)^n = \lambda^{\frac{(n-1)n}{2}}a^{-n}b^n = \lambda^{-\frac{n(n+1)}{2}}b^na^{-n}, \tag{8}$$

and similarly if b is invertible.

(3) *If either a or b is quasinilpotent, then ab is quasinilpotent also.*

(4) *If both a and b are quasinilpotent, then $a+b$ is quasinilpotent also.*

Proof. (1) The proof of (1) is induction.

(2) Substitute a^{-1} for a in (1).

(3) We consider two cases: $|\lambda| \leqslant 1$ and $|\lambda| > 1$. In the both of the cases, using the part (1), we get that $\|(ab)^n\|^{\frac{1}{n}} \to 0,\ n \to \infty$. Hence, ab is quasinilpotent.

(4) Let $c_1, c_2, \ldots, c_n$ be such that j of them are equal to a and $n-j$ are equal to b, for $0 \leqslant j \leqslant n$. Using that $c_1c_2\cdots c_n = \lambda^s b^{n-j}a^j$, for some $0 \leqslant s \leqslant j$, in the case when $|\lambda| \leqslant 1$ and that $(ab)^n = \lambda^{-k}a^jb^{n-j}$, for some $0 \leqslant k \leqslant n-j$, in the case when $|\lambda| \geqslant 1$, we have that

$$\|(a+b)^n\| \leqslant \sum_{j=0}^{n}\binom{n}{j}\|a^j\|\|b^{n-j}\|.$$

For $0 < \varepsilon < \min\{\|a\|, \|b\|\}$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geqslant n_0 \Rightarrow \|a^n\| \leqslant \varepsilon^n \quad \text{and} \quad \|b^n\| \leqslant \varepsilon^n.$$

Now, for $n \geqslant 2n_0$,

$$\begin{aligned}\|(a+b)^n\| &\leqslant \sum_{j=0}^{n} \binom{n}{j} \|a^j\|\|b^{n-j}\| \\ &= \left(\sum_{j=0}^{n_0} + \sum_{j=n_0+1}^{2n_0-1} + \sum_{j=2n_0}^{n}\right) \binom{n}{j} \|a^j\|\|b^{n-j}\| \\ &\leqslant (2\varepsilon)^n \max\left\{\frac{\|a\|}{\varepsilon}, \frac{\|b\|}{\varepsilon}\right\}^{n_0}.\end{aligned}$$

Hence, $a + b$ is quasinilpotent. $\square$

Lemma 2.2. *Let $a, b \in \mathcal{A}$ be such that a is invertible, b is quasinilpotent and $ab = \lambda ba$, $\lambda \neq 0$. Then $a - b$ is invertible and*

$$(a-b)^{-1} = \sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} a^{-(i+1)} b^i. \tag{9}$$

Proof. By Lemma 2.1, $a^{-1}b$ is quasinilpotent which implies that $1 - a^{-1}b$ is invertible. We have that

$$a - b = a(1 - a^{-1}b)$$

so

$$\begin{aligned}(a-b)^{-1} &= (1 - a^{-1}b)^{-1}a^{-1} \\ &= \left(\sum_{i=0}^{\infty} \lambda^{\frac{(i-1)i}{2}} a^{-i} b^i\right) a^{-1} \\ &= \sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} a^{-(i+1)} b^i,\end{aligned}$$

i.e. (9) holds. $\square$

Theorem 2.2. *Let $a, b \in \mathcal{A}^{\mathrm{d}}$. If $ab = \lambda ba$ and $ab \neq 0$, then*

(1) $a^{\mathrm{d}}b = \frac{1}{\lambda} ba^{\mathrm{d}}$,
(2) $ab^{\mathrm{d}} = \frac{1}{\lambda} b^{\mathrm{d}}a$.

Proof. (1) Denote aa^{d} by p. First, we will prove that p commutes with b. Since,

$$pb - pbp = pb(1-p) = p^n b(1-p) = (a^{\mathrm{d}})^n a^n b(1-p) = \lambda^n (a^{\mathrm{d}})^n b a^n (1-p),$$

we get that

$$\|pb - pbp\|^{\frac{1}{n}} \to 0, \quad n \to \infty,$$

i.e. $pb = pbp$. Similarly, we prove that $bp = pbp$. Hence, $pb = bp$.

Now,

$$ba^{\mathrm{d}} = bpa^{\mathrm{d}} = pba^{\mathrm{d}} = pbpa^{\mathrm{d}} = \lambda a^{\mathrm{d}}baa^{\mathrm{d}} = \lambda a^{\mathrm{d}}pb = \lambda a^{\mathrm{d}}b.$$

The proof of (2) is analogous. $\square$

The following theorem and Theorem 2.2 present a generalization of Theorem 2.3 from [3]:

Theorem 2.3. *Let $a, b \in \mathcal{A}^{\mathrm{d}}$. If $ab = \lambda ba$ and $ab \neq 0$, then*

$$(ab)^{\mathrm{d}} = b^{\mathrm{d}} a^{\mathrm{d}} = \frac{1}{\lambda} a^{\mathrm{d}} b^{\mathrm{d}}.$$

Proof. If $a^{\mathrm{d}} = 0$ then $a \in \mathcal{A}^{\mathrm{qnil}}$ and Lemma 2.1 (3) applies. If $a^{\mathrm{d}} \neq 0$, then $a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p$, for $p = 1 - a^{\pi}$, where a_1 is invertible in the algebra $\mathcal{A}_1 = p\mathcal{A}p$ and a_2 is quasinilpotent element of $\mathcal{A}_2 = (1-p)\mathcal{A}(1-p)$. Let $b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_p$. By Theorem 2.2, we have that b commutes with aa^{d}, so $b_{12} = 0$ and $b_{21} = 0$. Denote by $b_1 = b_{11}$ and $b_2 = b_{22}$. Since $ab = \lambda ba$, it follows that $a_i b_i = \lambda b_i a_i$, $i = 1, 2$.

Now,

$$a^{\mathrm{d}} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p \quad \text{and} \quad b^{\mathrm{d}} = \begin{bmatrix} b_1^{\mathrm{d}} & 0 \\ 0 & b_2^{\mathrm{d}} \end{bmatrix}_p.$$

Let us prove that

$$(a_1 b_1)^{\mathrm{d}} = b_1^{\mathrm{d}} a_1^{-1} = \frac{1}{\lambda} a_1^{-1} b_1^{\mathrm{d}}. \tag{10}$$

Since $a_1 b_1 = \lambda b_1 a_1$, we have that $b_1^{\mathrm{d}} = \left(\lambda a_1^{-1} b_1 a_1\right)^{\mathrm{d}} = \frac{1}{\lambda} a_1^{-1} b_1^{\mathrm{d}} a_1$. Hence, $b_1^{\mathrm{d}} a_1^{-1} = \frac{1}{\lambda} a_1^{-1} b_1^{\mathrm{d}}$.

Now, we prove that $(a_1 b_1)^{\mathrm{d}} = b_1^{\mathrm{d}} a_1^{-1}$ by a direct verification of (1) using the fact that $(a_1 b_1)^k = \lambda^{\frac{-(k-1)k}{2}} a_1^k b_1^k$.

By Lemma 2.1 (3), we have that $a_2 b_2$ is quasinilpotent, i.e. $(a_2 b_2)^{\mathrm{d}} = 0$.

Now, we conclude that

$$(ab)^{\mathrm{d}} = \begin{bmatrix} (a_1 b_1)^{\mathrm{d}} & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} b_1^{\mathrm{d}} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} \frac{1}{\lambda} a_1^{-1} b_1^{\mathrm{d}} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Hence, $(ab)^{\mathrm{d}} = b^{\mathrm{d}} a^{\mathrm{d}} = \frac{1}{\lambda} a^{\mathrm{d}} b^{\mathrm{d}}$. $\square$

Theorem 2.4. *Let $a, b \in \mathcal{A}^{\mathrm{d}}$. If $ab = \lambda ba$ and $\lambda \neq 0$, then $a - b$ is generalized Drazin invertible if and only if $c = aa^{\mathrm{d}}(a-b)bb^{\mathrm{d}}$ is generalized Drazin invertible. In this case,*

$$(a-b)^{\mathrm{d}} = c^{\mathrm{d}} + b^{\pi} \left(\sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} (a^{\mathrm{d}})^{i+1} b^i \right) - \left(\sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} a^i (b^{\mathrm{d}})^{i+1} \right) a^{\pi}. \tag{11}$$

Proof. As in the proof of Theorem 2.3, we have that $a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p$ for $p = 1 - a^{\pi}$, where a_1 is invertible in the algebra $p\mathcal{A}p$ and a_2 is quasinilpotent element of $(1-p)\mathcal{A}(1-p)$. Also, $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p$ and $a_i b_i = \lambda b_i a_i$, $i = 1, 2$.

Let us suppose that $b_2^{\mathrm{d}} \neq 0$ (the case when $b_2^{\mathrm{d}} = 0$ is trivial). Then we have that $b_2 = \begin{bmatrix} b_2' & 0 \\ 0 & b_2'' \end{bmatrix}_{p'}$ relative to $p' = 1 - b_2^{\pi}$, where b_2' is invertible in the algebra $p'\mathcal{A}p'$ and b_2'' is quasinilpotent element of $\mathcal{A}$. Also, $a_2 = \begin{bmatrix} a_2' & 0 \\ 0 & a_2'' \end{bmatrix}_{p'}$, where a_2', a_2'' are quasinilpotent and $a_2' b_2' = \lambda b_2' a_2'$.

First, we will show that $a_2 - b_2$ is generalized Drazin invertible. From Lemma 2.2, we conclude that $a_2' - b_2'$ is invertible in subalgebra $p'\mathcal{A}p'$ and

$$(a_2' - b_2')^{-1} = -\sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} (a_2')^i (b_2')^{-(i+1)}.$$

Since a_2'' and b_2'' are quasinilpotent, we deduce that $a_2'' - b_2''$ is generalized Drazin invertible and $(a_2'' - b_2'')^{\mathrm{d}} = 0$ (by Lemma 2.1 (4)). Hence, $a_2 - b_2$ is generalized Drazin invertible by Theorem 2.1.

Now, the generalized Drazin invertibility of $a - b$ is equivalent to the generalized Drazin invertibility of $a_1 - b_1$.

As in the first part of the proof, let us suppose that $b_1^{\mathrm{d}} \neq 0$ (the case when $b_1^{\mathrm{d}} = 0$ is trivial). Then, $b_1 = \begin{bmatrix} b_1' & 0 \\ 0 & b_1'' \end{bmatrix}_q$ relative to $q = 1 - b_1^{\pi}$, where b_1' is invertible in the algebra $q\mathcal{A}q$ and b_1'' is quasinilpotent element of $(1-q)\mathcal{A}(1-q)$. Similarly, we get that $a_1 = \begin{bmatrix} a_1' & 0 \\ 0 & a_1'' \end{bmatrix}_q$, where a_1', a_1'' are invertible in subalgebra $q\mathcal{A}q$ and $(1-q)\mathcal{A}(1-q)$, respectively and $a_1'b_1' = \lambda b_1'a_1'$, $a_1''b_1'' = \lambda b_1''a_1''$. By Lemma 2.2, we have that $a_1'' - b_1''$ is invertible in subalgebra $(1-q)\mathcal{A}(1-q)$ and

$$(a_1'' - b_1'')^{-1} = \sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} (a_1'')^{-(i+1)} (b_1'')^i.$$

Hence, generalized Drazin invertibility of $a - b$ is equivalent to generalized Drazin invertibility of $a_1' - b_1'$.

Using the representation introduced above, we can conclude that the generalized Drazin invertibility of c is equivalent to the generalized Drazin invertibility of $a_1' - b_1'$ also. Hence, $a - b$ is generalized Drazin invertible if and only if $c = aa^{\mathrm{d}}(a-b)bb^{\mathrm{d}}$ is generalized Drazin invertible.

The formula (11) follows easily by the calculation using the above representations. □

Theorem 2.5. *Let $a, b \in \mathcal{A}^{\mathrm{d}}$ be such that $ab = \lambda ba$ and $\lambda \neq 0$.*

(1) *If $\sigma(ba^{\mathrm{d}}) = \{0\}$, then $a - b$ is generalized Drazin invertible and*

$$(a-b)^{\mathrm{d}} = b^{\pi} \left(\sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} (a^{\mathrm{d}})^{(i+1)} b^i \right) - \left(\sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} a^i (b^{\mathrm{d}})^{(i+1)} \right) a^{\pi}. \tag{12}$$

(2) *If $1 \notin \sigma(ba^{\mathrm{d}})$, then $a - b$ is generalized Drazin invertible and*

$$(a-b)^{\mathrm{d}} = a^{\mathrm{d}}(1 - ba^{\mathrm{d}})^{-1} bb^{\mathrm{d}} + b^{\pi} \left(\sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} (a^{\mathrm{d}})^{(i+1)} b^i \right) - \left(\sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} a^i (b^{\mathrm{d}})^{(i+1)} \right) a^{\pi}. \tag{13}$$

(3) *If $1 \in \sigma(ba^{\mathrm{d}})$, $d = 1 - ba^{\mathrm{d}}$ is generalized Drazin invertible,* $\mathrm{ind}(d) = 1$ *and $a^2 b^{\mathrm{d}} dd^{\#} = d^{\#} da^2 b^{\mathrm{d}}$, then $a - b$ is generalized Drazin invertible and*

$$(a-b)^{\mathrm{d}} = a^{\mathrm{d}} d^{\#} dab^{\mathrm{d}} dd^{\#} ba^{\mathrm{d}} + a^{\mathrm{d}} d^{\#} ba^{\mathrm{d}} bb^{\mathrm{d}} + b^{\pi} \left(\sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} (a^{\mathrm{d}})^{(i+1)} b^i \right) - \left(\sum_{i=0}^{\infty} \lambda^{\frac{i(i+1)}{2}} a^i (b^{\mathrm{d}})^{(i+1)} \right) a^{\pi}. \tag{14}$$

Proof. (1) If $\sigma(ba^{\mathrm{d}}) = \{0\}$, using the representations introduced in the proof of Theorem 2.3 and some spectral properties, we get that $b_1^{\mathrm{d}} = 0$. Now, $c = aa^{\mathrm{d}}(a-b)bb^{\mathrm{d}} = 0$ and from (11), we get that (12) holds.

(2) If $1 \notin \sigma(ba^{\mathrm{d}})$, then $1 - ba^{\mathrm{d}}$ is invertible. Using the representations introduced in the proof of Theorem 2.3, we get that $a_1 - b_1$ is invertible in algebra $p\mathcal{A}p$. Now,

$$c^{\mathrm{d}} = \begin{bmatrix} \left((a_1 - b_1)b_1 b_1^{\mathrm{d}}\right)^{\mathrm{d}} & 0 \\ 0 & 0 \end{bmatrix}_p \quad \text{and} \quad a^{\mathrm{d}}(1 - ba^{\mathrm{d}})^{-1}bb^{\mathrm{d}} = \begin{bmatrix} (a_1 - b_1)^{-1}b_1 b_1^{\mathrm{d}} & 0 \\ 0 & 0 \end{bmatrix}_p .$$

Since a_1 commutes with $b_1 b_1^{\mathrm{d}}$, $c^{\mathrm{d}} = a^{\mathrm{d}}(1 - ba^{\mathrm{d}})bb^{\mathrm{d}}$. Now, from (11), we get that (13) holds.

(3) Using the representations introduced in the proof of Theorem 2.3 and straightforward calculations, we get that

$$c^{\mathrm{d}} = a^{\mathrm{d}}d^{\#}dab^{\mathrm{d}}dd^{\#}ba^{\mathrm{d}} + a^{\mathrm{d}}d^{\#}ba^{\mathrm{d}}bb^{\mathrm{d}}.$$

From (11), we get that (14) holds. $\square$

Remark. If $ab = \lambda ba$, it follows that $\sigma(ab) = \lambda\sigma(ba)$. Now,

$$\sigma(ba) \cup \{0\} = \lambda\sigma(ba) \cup \{0\}.$$

When $\lambda^2 \neq 1$, set $\sigma(ab)$ is finite if and only if $\sigma(ab) = \{0\}$, i.e. ab is quasinilpotent.

Theorem 2.6. *Let $a, b \in \mathcal{A}$ be such that b and $1 + b$ are generalized Drazin invertible, a is invertible and $ba = b$.*

(1) *If $b(1+b)^{\pi} = 0$, then $1 + b$ is invertible and*

$$(a+b)^{-1} = a^{-1}(1+b)^{-1} = a^{-1} - a^{-1}b(1+b)^{-1}.$$

(2) *$a + b$ is generalized Drazin invertible if and only if $1 + b$ is generalized Drazin invertible. Moreover, we have* $\operatorname{ind}(1+b) = \operatorname{ind}(a+b)$, $(ab)^{\mathrm{d}} = ab^{\mathrm{d}}$,

$$(a+b)^{\mathrm{d}} = a^{-1}(1+b)^{\mathrm{d}} + \left(\sum_{i=0}^{\infty} a^{-(i+2)}b^{\pi}a(1+b)^i\right)(1+b)^{\pi} \tag{15}$$

and

$$(a+b)(a+b)^{\mathrm{d}} = (1+b)(1+b)^{\mathrm{d}} + (1+b)^{\mathrm{d}}\left(\sum_{i=0}^{\infty} a^{-(i+1)}b^{\pi}a(1+b)^i\right)(1+b)^{\pi}. \tag{16}$$

Proof. (1) From the generalized Drazin invertibility of b, we have that $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p$, for $p = 1 - b^{\pi}$, where b_1 is invertible in algebra $p\mathcal{A}p$ and b_2 is quasinilpotent element of $\mathcal{A}$. Now, since b_2 is quasinilpotent, it follows that $1 + b_2$ is invertible in algebra $(1-p)\mathcal{A}(1-p)$ and $b(1+b)^{\pi} = \begin{bmatrix} b_1(1+b_1)^{\pi} & 0 \\ 0 & 0 \end{bmatrix}_p$. Hence, $b_1(1+b_1)^{\pi} = 0$. From the invertibility of b_1, we conclude that $1 + b_1$ is invertible in the algebra $p\mathcal{A}p$, so $1 + b$ is invertible in $\mathcal{A}$. Now,

$$(a+b)^{-1} = ((1+b)a)^{-1} = a^{-1}(1+b)^{-1} = a^{-1} - a^{-1}b(1+b)^{-1}.$$

(2) As in the first part of the proof, we have that $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p$, for $p = 1 - b^{\pi}$, where b_1 is invertible in algebra $p\mathcal{A}p$ and b_2 is quasinilpotent element of $\mathcal{A}$. Let $a = \begin{bmatrix} a_1 & a_3 \\ a_4 & a_2 \end{bmatrix}_p$. From $ba = b$, we have that $a_1 = p$, $a_3 = 0$, $b_2a_4 = 0$ and $b_2a_2 = b_2$. Now, $a + b = \begin{bmatrix} 1 + b_1 & 0 \\ a_4 & a_2 + b_2 \end{bmatrix}_p$. Since $a_2 + b_2 = (1+b_2)a_2$ and $1 + b_2 \in ((1-p)\mathcal{A}(1-p))^{-1}$, we have that $a_2 + b_2$ is invertible and

$$\begin{aligned}(a_2+b_2)^{-1} &= a_2^{-1}(1+b_2)^{-1}\\ &= a_2^{-1} - a_2^{-1}(1+b_2)^{-1}b_2.\end{aligned} \tag{17}$$

So the generalized Drazin invertibility of $a+b$ is equivalent to the generalized Drazin invertibility of $1+b_1$ which is equivalent to the generalized Drazin invertibility of $1+b$. Furthermore, $\mathrm{ind}(1+b) = \mathrm{ind}(a+b) = \mathrm{ind}(1+b_1)$ (it could be ∞).

Using Theorem 2.1, we get that

$$(ab)^{\mathrm{d}} = \left(\begin{bmatrix} b_1 & 0 \\ a_4 b_1 & a_2 b_2 \end{bmatrix}_p\right)^{\mathrm{d}} = \begin{bmatrix} b_1^{-1} & 0 \\ a_4 b_1^{-1} & 0 \end{bmatrix}_p = ab^{\mathrm{d}}.$$

Also, by Theorem 2.1 and (17),

$$(a+b)^{\mathrm{d}} = \left(\begin{bmatrix} 1+b_1 & 0 \\ a_4 & a_2+b_2 \end{bmatrix}_p\right)^{\mathrm{d}} = \begin{bmatrix} (1+b_1)^{\mathrm{d}} & 0 \\ u & (a_2+b_2)^{-1} \end{bmatrix}_p,$$

where

$$\begin{aligned} u &= \sum_{n=0}^{\infty} \left((a_2+b_2)^{-1}\right)^{n+2} a_4(1+b_1)^n(1+b_1)^{\pi} - (a_2+b_2)^{-1}a_4(1+b_1)^{\mathrm{d}} \\ &= \sum_{n=0}^{\infty} \left(a_2^{-1}\right)^{n+2} a_4(1+b_1)^n(1+b_1)^{\pi} - a_2^{-1}a_4(1+b_1)^{\mathrm{d}}. \end{aligned}$$

Since,

$$a^{-1}(1+b)^{\mathrm{d}} = \begin{bmatrix} (1+b_1)^{\mathrm{d}} & 0 \\ -a_2^{-1}a_4(1+b_1)^{\mathrm{d}} & (a_2+b_2)^{-1} \end{bmatrix}_p$$

and

$$\left(\sum_{n=0}^{\infty} a^{-(n+2)} b^{\pi} a(1+b)^n\right)(1+b)^{\pi} = \begin{bmatrix} 0 & 0 \\ \sum_{n=0}^{\infty} a_2^{-(n+2)} a_4(1+b_1)^n(1+b_1)^{\pi} & 0 \end{bmatrix}_p$$

we get that (15) holds. Similarly, we get (16). □

Using the same method of proof as in Theorem 2.6, we can prove the following result:

Theorem 2.7. *Let $a, b \in \mathcal{A}^{\mathrm{d}}$ be such that $aa^{\mathrm{d}}b$ and $aa^{\mathrm{d}}(1+b)$ are generalized Drazin invertible. If $ab = aba$, then $a+b$ is generalized Drazin invertible if and only if $1+b$ is generalized Drazin invertible. Moreover, we have $(ab)^{\mathrm{d}} = a^{\pi}aba^{\mathrm{d}}(b^{\mathrm{d}})^2 + a^2a^{\mathrm{d}}b^{\mathrm{d}}$, and*

$$\begin{aligned}(a+b)^{\mathrm{d}} = {} & A^{\mathrm{d}} + B^{\mathrm{d}} - B^{\mathrm{d}}bA^{\mathrm{d}} + \left(\sum_{n=0}^{\infty}(B^{\mathrm{d}})^{n+2}baa^{\mathrm{d}}(a+b)^n\right) \\ & \times \left(aa^{\mathrm{d}}(1+b)^{\pi} - aa^{\mathrm{d}}(1+b)\left(\sum_{n=0}^{\infty}(a^{\mathrm{d}})^{n+1}b^{\pi}a(1+b)^{\pi}\right)\right. \\ & + \left(a^{\pi} - bb^{\mathrm{d}}\left(\sum_{n=0}^{\infty}(b^{\mathrm{d}})^n a^n\right)a^{\pi}\right) \\ & \times \left(\sum_{n=0}^{\infty}(a+b)^n a^{\pi} b(A^{\mathrm{d}})^{n+2}\right),\end{aligned}$$

where

$$A^{\mathrm{d}} = a^{\mathrm{d}}(1+b)^{\mathrm{d}} + \left(\sum_{n=0}^{\infty}(a^{\mathrm{d}})^{n+2}b^{\pi}a(1+b)^{n}\right)aa^{\mathrm{d}}(1+b)^{\pi}$$

and

$$B^{\mathrm{d}} = b^{\mathrm{d}}\left(\sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n}a^{n}\right)a^{\pi}.$$

Using Corollary 2.3 [2], we get the following result:

Theorem 2.8. *Let $a, b \in \mathcal{A}$ be generalized Drazin invertible. If $a^{\mathrm{d}}b = 0$ and $aba^{\pi} = 0$, then*

$$(a+b)^{\mathrm{d}} = a^{\mathrm{d}} + B^{\mathrm{d}} - B^{\mathrm{d}}ba^{\mathrm{d}} + \left(a^{\pi} - bb^{\mathrm{d}}\left(\sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n}a^{n}\right)a^{\pi}\right) \times \left(\sum_{n=0}^{\infty}(a+b)^{n}a^{\pi}b(a^{\mathrm{d}})^{n+2}\right),$$

where

$$B^{\mathrm{d}} = b^{\mathrm{d}}\left(\sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n}a^{n}\right)a^{\pi}.$$

The following corollary is a generalization of Theorem 2.1 [6] for matrices and Theorem 2.3 [4] for bounded operators:

Corollary 2.1. *Let $a, b \in \mathcal{A}$ be generalized Drazin invertible. If $ab = 0$, then*

$$(a+b)^{\mathrm{d}} = b^{\mathrm{d}}\left(\sum_{n=0}^{\mathrm{ind}(a)-1}(b^{\mathrm{d}})^{n}a^{n}\right)a^{\pi} + b^{\pi}\left(\sum_{n=0}^{\mathrm{ind}(b)-1}b^{n}(a^{\mathrm{d}})^{n}\right)a^{\mathrm{d}}.$$

Acknowledgment

The author would like to thank the referee for reading carefully the first version of this note and made constructive suggestions for its improvement. Supported by Grant No. 144003 of the Ministry of Science, Technology and Development, Republic of Serbia.

References

[1] N. Castro González, J.J. Koliha, New additive results for the g-Drazin inverse, Proc. Roy. Soc. Edinburgh Sect. A, 134 (2004) 1085–1097.
[2] D.S. Cvetković-Ilić, D.S. Djordjević, Y. Wei, Additive results for the generalized Drazin inverse in a Banach algebra, Linear Algebra Appl. 418 (2006) 53–61.
[3] C.Y. Deng, The Drazin inverse of bounded operators with commutativity up to a factor, Appl. Math. Comput. 206 (2008) 695–703.
[4] D.S. Djordjević, P.S. Stanimirović, On the generalized Drazin inverse and generalized resolvent, Czechoslovak Math. J. 51 (126) (2001) 617–634.
[5] M.P. Drazin, Pseudoinverse in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506–514.
[6] R.E. Hartwig, G. Wang, Y. Wei, Some additive results on Drazin inverse, Linear Algebra Appl. 322 (2001) 207–217.
[7] R.E. Harte, Spectral projections, Irish Math. Soc. Newslett. 11 (1984) 10–15.
[8] R.E. Harte, Invertibility and Singularity for Bounded Linear Operators, Marcel Dekker, New York, 1988.
[9] R.E. Harte, On quasinilpotents in rings, Panamer. Math. J. 1 (1991) 10–16.
[10] J.J. Koliha, A generalized Drazin inverse, Glasg. Math. J. 38 (1996) 367–381.
[11] C.D. Meyer Jr., N.J. Rose, The index and the Drazin inverse of block triangular matrices, SIAM J. Appl. Math. 33 (1) (1977) 1–7.